Alessandro Masullo's Proof of the Riemann Hypothesis

An overview of the proposed proof of the Generalized Riemann Hypothesis in *The Music of the Primes*

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On the same Friday in November 2026, two proofs of the Generalized Riemann Hypothesis appeared. One was Levent Alpöge's, found with Claude and released as seventeen million lines of Lean. The other was Alessandro Maria Masullo's *The Music of the Primes*: a 940-page manuscript, twelve years in the making, released as an unfinished draft and announced as "all human, with no AI." The two arguments have almost nothing in common. Alpöge's proves a one-sided inequality about primes using arithmetic geometry on higher-rank groups. Masullo's proves an identity, a trace formula in the tradition of Weil, Connes and Meyer, using functional analysis on the simplest group of all, GL_1.

This document is a companion to the overview of Alpöge's proof. It explains how Masullo's manuscript came out, what it claims, and how its strategy works. It then separates three kinds of problem: errors that have been found and can be repaired, parts of the manuscript that are not yet written, and the one step on which everything depends. That step is a trace formula which is *equivalent* to the Generalized Riemann Hypothesis, and in the current draft its proof is incomplete.

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What Happened

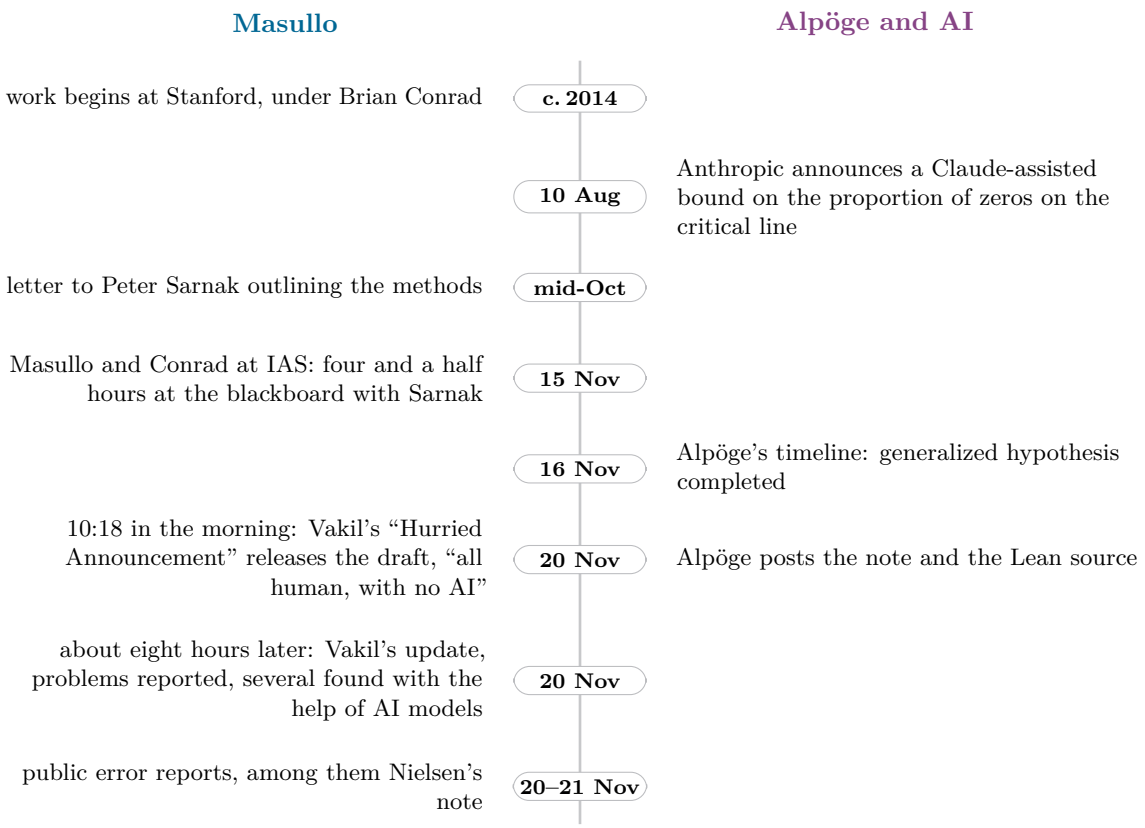
Masullo began the work, by his own account, near the end of his first year as a graduate student at Stanford under Brian Conrad. The manuscript's acknowledgements say it "took nearly 12 years to assemble" [1, Acknowledgements]. Conrad and Ravi Vakil were arranging for Masullo to spend a year at Stanford polishing the exposition before any announcement. In mid-October 2026 Masullo sent Peter Sarnak a long letter outlining the methods, and on the Sunday before the release, 15 November, Masullo and Conrad visited Sarnak at the Institute for Advanced Study for what the acknowledgements describe as "a four and a half hour long blackboard discussion."

On Friday 20 November 2026 Levent Alpöge, a mathematician at Anthropic, posted a short note [3], together with the Lean source of a formal proof of the Riemann Hypothesis and its generalization to Dirichlet L-functions. The note's timeline puts the completion of the generalized hypothesis on 16 November. The autumn had already brought AI-assisted results about the zeta function: in August, Anthropic had announced a Claude-assisted improvement in the proportion of zeros known to lie on the critical line [4].

The same day, at 10:18 in the morning, Vakil posted "Hurried Announcement: Alessandro Masullo GRH" on his blog, with the manuscript attached [2]. Vakil wrote from a meeting of the American Mathematical Society's Executive Committee and Board of Trustees. The post explains the change of plan: "The hope was to have Brian work with Alessandro for a year to polish the writing and then announce it after running it by enough people. Now it seems best to put out the current draft." It reports that Conrad "has been looking closely at it for months and gave extensive feedback which is being incorporated, and it is so far looking good to him," and adds: "It is all human, with no AI." Readers were asked to "quickly check it, formalize, fill any small gaps if they occur."

Masullo's acknowledgements give the same reason in his own words [1, Acknowledgements]: "In light of the current climate I am being advised to release the current version as it is. The math is complete, and only filling in some writing remains on several parts across this long paper." They are candid about the state of the draft. The introduction "is unfinished and will be soon replaced by a proper one," and Masullo and Conrad "are aware of several glitches here and there." They end with a message to mathematicians "who feel discouraged by the pace and tone of the current climate": "we are still here. We are very much alive. We are not going anywhere."

Neither the post nor the manuscript names Alpöge. The timing is nonetheless the context of the release: a human proof that was to be held back for a year of polishing was put out on the day an AI-assisted one appeared, and the press reported the two side by side [5].



The release, not to scale. Masullo's events are on the left, Alpöge's and Anthropic's on the right. Dates for Masullo come from the manuscript's acknowledgements and Vakil's post.

Problems surfaced at once. About eight hours after the announcement, Vakil updated the post [2]: "Several people have contacted me to report problems in the proposed proof," several of them "identified with the assistance of different AI models," and "all issues need to be resolved to the satisfaction of the community before this proof would be accepted as true. (I make no claim as to the correctness of the proof strategy.)" Within a day there were public reports locating specific errors. A three-page note by Jenny Lorraine Nielsen, "A Failure of Uniform Boundedness in Masullo's Proposed Proof of the Generalized Riemann Hypothesis" [6], gave two counterexamples, and an issue in a public formalization repository [7] and an independent research log [8] pinned further errors to particular pages.

This document is written from a reading of the manuscript as released on 20 November. It describes the argument as the manuscript presents it and checks the reported errors against the text. It distinguishes what the draft proves, what it states without proof, and what it does not yet contain. It can be read at two depths. For the big picture, read the plain account of the strategy, the comparison with Alpöge's proof and the final assessment. For the details, read the account of the argument and the sections on errors and gaps, which cite the manuscript by section and page. It is not a referee report. A revision may close any of the gaps described here, and none of what follows should be read as a judgement on the final form of the work.

What Is Claimed

Riemann's zeta function is the first of a family of L-functions, one for each *Hecke character*. Fix a number field F, a finite extension of $\mathbb{Q}$. Its idele class group $C_F = \mathrm{GL}_1(\mathbb{A}_F)/F^\times$ is the group that class field theory attaches to F, and a Hecke character is a continuous unitary character χ of C_F. For $F = \mathbb{Q}$ these include the Dirichlet characters, and the trivial character gives ζ itself. Each χ has an L-function with an Euler product over the prime ideals $\mathfrak{p}$ of F,

$$L(s,\chi) = \prod_{\mathfrak{p}} \frac{1}{1 - \chi(\mathfrak{p})\, N\mathfrak{p}^{-s}},$$

which converges for $\operatorname{Re} s > 1$. It extends to the whole plane and satisfies a functional equation exchanging s and $1 - s$. The manuscript's main claim is that every such function has its nontrivial zeros on the line of symmetry.

Claimed Theorem (Generalized Riemann Hypothesis for Hecke L-functions)**.** Let F be a number field and χ a unitary Hecke character of F. Then every zero of $L(s,\chi)$ in the strip $0 < \operatorname{Re} s < 1$ lies on the line $\operatorname{Re} s = \frac{1}{2}$.

In one respect this is broader than Alpöge's theorem: it covers every number field and every Hecke character, while Alpöge's covers Dirichlet characters over $\mathbb{Q}$. In another it is narrower: it concerns only GL_1, the abelian case, and makes no claim about L-functions of automorphic forms on larger groups. The two claims overlap exactly for $F = \mathbb{Q}$. The manuscript also contains a second, stronger claim, a bound on the multiplicity of zeros, which is discussed under the gaps below.

The Strategy in Plain Terms

Zeros as a spectrum

In 1952 Weil found an exact identity [9] relating primes and zeros, the *explicit formula*. Take a test function φ on C_F. On one side of the identity is a sum over the places of F, the primes and the archimedean places, each contributing a local term $W_v(\varphi)$ that depends only on φ near the corresponding prime powers. On the other side is a sum over all zeros ρ of $L(s,\chi)$, each weighted by a Mellin transform of φ:

$$\sum_v W_v(\varphi) \;=\; \mathcal{M}\varphi(|\cdot|^{1/2}) + \mathcal{M}\varphi(|\cdot|^{-1/2}) \;-\; \sum_{\text{all zeros } \rho} N(\rho)\, \mathcal{M}\varphi(\chi|\cdot|^{\rho-1/2}).$$

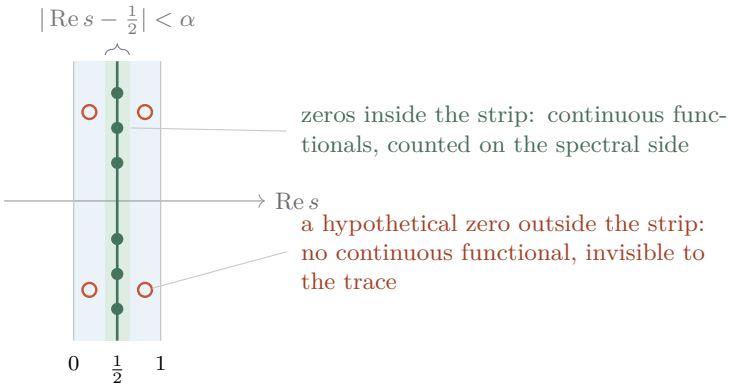
Here $N(\rho)$ is the multiplicity of the zero and $\mathcal{M}\varphi(c) = \int_{C_F} \varphi(g)c(g)\, d\mu(g)$.

The identity has the shape of a trace formula, like Selberg's for a hyperbolic surface, in which closed geodesics on one side match eigenvalues on the other. Connes proposed in the 1990s to take that resemblance literally [12]: find a space of functions on which the idele class group acts, compute the trace of the action in two ways, and read off the zeros as a spectrum. The hope was that the right space would make the location of the zeros visible, as eigenvalues of a unitary operator are forced onto the unit circle.

The difficulty is that in the natural space, every zero appears on the spectral side, wherever it lies. Ralf Meyer carried out the trace calculation rigorously [13] in a space of rapidly decaying functions, and the result was Weil's explicit formula exactly. That is a theorem, but it says nothing about where the zeros are.

Masullo's move: make the wrong zeros invisible

Masullo works in a space whose topology is too coarse to see zeros away from the critical line. Its norm weights a function by $\max(|g|^{2\alpha}, |g|^{-2\alpha})$, for a fixed $\alpha > 0$ that can be as small as one likes. In that space, the functional "evaluate the Mellin transform at ρ" is continuous only when ρ lies in the strip $|\operatorname{Re}\rho - \frac{1}{2}| < \alpha$. A zero outside the strip has no continuous functional attached to it, so it cannot appear in a trace. Masullo calls the zeros "victims" of the topology [1, §1.7.2].



What the coarse topology sees. Only zeros in the green band contribute to the spectral side of Masullo's trace formula. Off-line zeros (red rings), if they existed, would drop out.

The whole proof is then a comparison. If one could show that the *geometric* side of the trace, the sum over places, is the same in the coarse space as in Meyer's fine one, the two formulas would agree on the left and differ on the right only by the zeros outside the strip. Those zeros would therefore contribute nothing for every admissible test function, so there would be none. Since α is arbitrary, every zero would lie on the line.

> **The whole proof is one identity**
>
> Masullo's main theorem (Theorem 12.5.1.1) is a trace formula in which the place-by-place side is Weil's and the zero side runs only over zeros in the strip $|\operatorname{Re}\rho - \frac{1}{2}| < \alpha$. Connes observed in 1999 [12] that a global trace formula of this kind is *equivalent* to the Generalized Riemann Hypothesis for the L-functions involved. So the trace formula is not a step toward the result; it is the result. Everything in the manuscript is in service of proving it, and an assessment of the manuscript is an assessment of that proof.

How the geometric side is computed

The obstacle is the infinitely many primes. In Meyer's fine topology, the passage from the global space to a finite set S of places is made by infinite Euler-product operators, which converge because the fine topology controls functions on the half-plane $\operatorname{Re} s > 1$, where Euler products converge absolutely. In a coarse topology tied to the critical strip they do not converge. Masullo identifies this as the central difficulty and writes of it: "Mertens' Theorem is, in my eyes, a herald for GRH. If you clash against it, it means that you are entering forbidden territory" [1, §1.8.1].

His answer is to degenerate one prime at a time instead of all at once. At each step he removes the Euler factor at one prime, keeps exact track of what was lost, and computes the trace of the lost

piece directly. Each lost piece turns out to carry a basis of exponentials, like a Fourier basis on a circle, and Poisson summation shows that its trace vanishes for all but finitely many primes. What remains after all the primes have been removed is a finite-S problem, which Connes solved in 1998. If this bookkeeping is valid in the coarse topology, the geometric side comes out as Weil's, and the proof is complete.

The Argument

This section follows the manuscript's argument in order. The steps are stated as the manuscript states them; the next sections say which of them are complete.

Spaces of theta functions

The objects are functions on the idele class group built from Schwartz–Bruhat functions on the adeles. For f on $\mathbb{A}_F$ with $f(0) = \hat{f}(0) = 0$, the *theta function* $\Phi_F(f)(g) = \sum_{\xi \in F^\times} f(\xi g)$ is a rapidly decaying function on C_F. Its Mellin transform is the zeta integral of Tate's thesis, and so it is divisible by $L(s, \chi)$. Chapters 2 to 6 establish the topological structure of these spaces in great generality, in five "Structure Theorems." The upshot, for each character χ, is the *divisor exact sequence*, which under the Mellin transform reads

$$0 \longrightarrow L(s + \tfrac{1}{2}, \chi) \cdot \mathcal{AS} \longrightarrow \mathcal{AS} \longrightarrow \prod_{L(\rho,\chi)=0} \mathbb{C}^{N(\rho)} \longrightarrow 0.$$

Here $\mathcal{AS}$ is a space of entire functions that are rapidly decreasing on vertical strips. The theta functions form the subspace of functions divisible by L, and the quotient records the value and the first $N(\rho) - 1$ derivatives at each zero. The name comes from the sequence $0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$ of a divisor D on a complex manifold. Here the divisor is the divisor of zeros of L.

The coarse completion and the spectral side

For $I = [-\alpha, \alpha]$ and a Sobolev degree $s \geq 0$, the *Hardy–Sobolev space* $W_I^{s,2}(C_F)$ is the completion in the norm

$$\|\phi\|_{s,I}^2 = \int_{C_F} |\phi(g)|^2 \, \max\{|g|^{-2\alpha}, |g|^{2\alpha}\} \, (1 + |\log |g||^2)^{s/2} \, d\mu(g).$$

Under the Mellin transform it becomes a space of holomorphic functions on the strip $|\operatorname{Re} z| < \alpha$, with boundary conditions on its edges. Completing the divisor sequence in this norm changes its right-hand end. The closure of the multiples of L is now the space of functions vanishing at the zeros *inside* the strip, so the quotient sees only those zeros.

A test function φ on C_F acts by convolution. On the quotient its matrix, in the basis of point evaluations and their derivatives, consists of Jordan blocks with diagonal entries $\mathcal{M}\varphi(\chi|\cdot|^{\rho-1/2})$. The operator is nuclear, so its trace is the sum of those entries:

$$\operatorname{Tr}\big(A(\varphi) \text{ on the quotient}\big) = \sum_{|\operatorname{Re}\rho - \frac{1}{2}| < \alpha} N(\rho) \, \mathcal{M}\varphi(\chi|\cdot|^{\rho-1/2}).$$

This is the spectral side, established in Chapter 10. Masullo defines a *spectral kernel* representing this operator and a *geometric kernel*

$$k_{\mathrm{geom}}(x,y) = \varphi(xy^{-1}) - k_{\mathrm{spec}}(x,y),$$

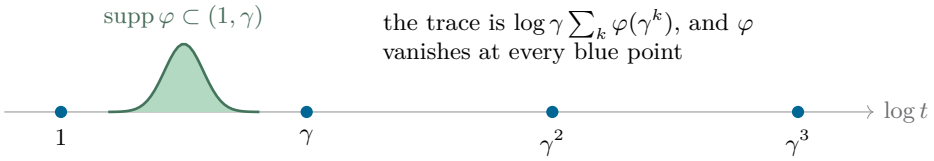
the convolution kernel minus the spectral kernel. The test function is required to vanish on the norm-one classes, so the convolution kernel vanishes on the diagonal. The integral of k_{geom} along the diagonal is therefore minus the spectral side, by construction. The entire manuscript is about computing that diagonal integral a second way.

The toy model

The mechanism that is meant to compute it is visible in a one-dimensional example. Fix $\gamma > 1$, standing in for the norm of a prime, and consider functions on the circle $\mathbb{R}_{>0}/\gamma^{\mathbb{Z}}$. They have a Schauder basis of exponentials $t \mapsto \exp(2\pi i m \log t/\log\gamma)$, and the convolution operator by a test function φ on $\mathbb{R}_{>0}$ multiplies the m-th basis vector by $\mathcal{M}\varphi(2\pi i m/\log\gamma)$. Its trace, by Poisson summation, is

$$\sum_{m\in\mathbb{Z}} \mathcal{M}\varphi\Big(\frac{2\pi i m}{\log\gamma}\Big) = \log\gamma \sum_{k\in\mathbb{Z}} \varphi(\gamma^k).$$

If φ is supported strictly inside $(1,\gamma)$, it misses every point γ^k, and the trace is zero.



The toy model behind the vanishing of the orbital traces. A test function whose support fits between consecutive powers of γ has zero trace on the circle $\mathbb{R}_{>0}/\gamma^{\mathbb{Z}}$. In the proof, γ is the norm of a prime, so a fixed test function misses all but finitely many primes.

Peeling off the primes

The global analogue replaces $\mathbf{1}_{\mathcal{O}_v}$ by $\mathbf{1}_{\mathcal{O}_v^\times}$ in the local component of f at one prime v outside a fixed finite set S. Under the Mellin transform this multiplies the zeta integral by $1 - \chi(\mathfrak{p}_v)N\mathfrak{p}_v^{-s}$, the reciprocal of one Euler factor. Doing this for all places above the first k rational primes defines a decreasing filtration of the theta space, and a matching *divisorial stratification* of the ideles by the sets where the components above those primes are units.

Each step of the filtration has a quotient which is a twisted copy of the toy model at $\gamma = N\mathfrak{p}_v$, and so carries its own exponential basis, an *orbital Schauder basis*. Chopping the geometric kernel along the strata with the indicator functions α_k of the strata gives pieces $k_p = (\alpha_{k+1} - \alpha_k)k_{\mathrm{geom}}$, one for each prime p. Theorem 11.3.2.1 computes the diagonal integral of each piece with the orbital basis. The answer is Weil's local term at the places above p, which is zero when the support of φ avoids the powers of $N\mathfrak{p}_v$. Taking S large enough, every piece outside S contributes zero.

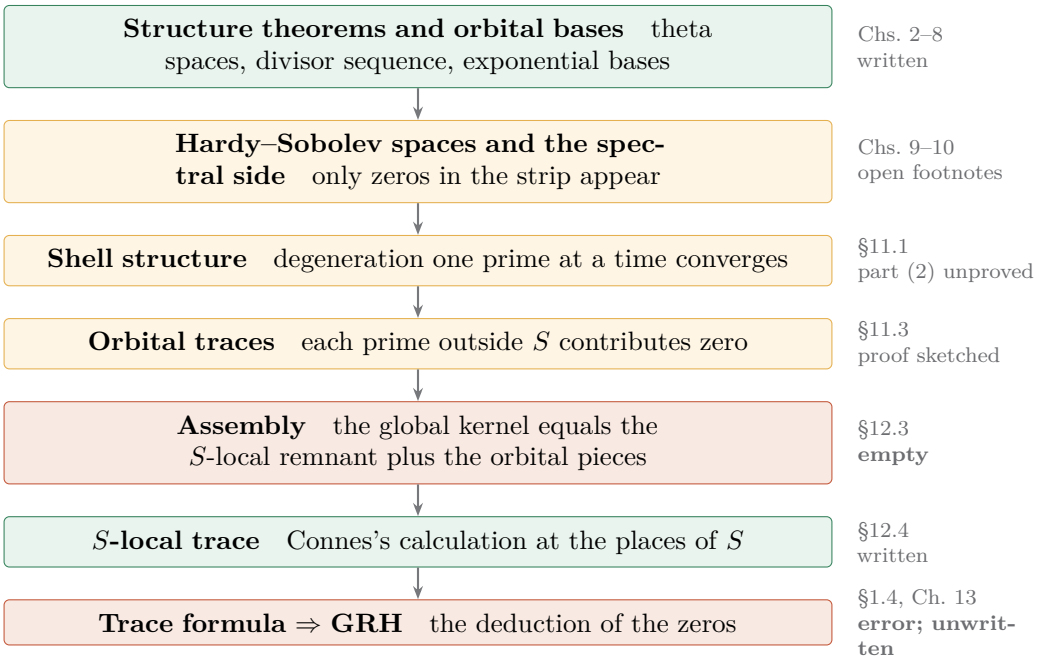
The finite remnant and the conclusion

After all the primes outside S have been peeled off, what is left of the kernel lives on the S-ideles. Its diagonal integral is computed in Theorem 12.4.1.1 by a version of Connes's 1998 S-local calculation [12, §VII], which gives Weil's local terms at the places of S. Adding the pieces gives the main theorem.

Claimed Theorem (Trace Formula, Theorem 12.5.1.1)**.** Let φ be a smooth compactly supported function on C_F vanishing on the norm-one classes C_F^1, and $I = [-\alpha, \alpha]$. Then

$$\sum_v W_v(\varphi) \;=\; \mathcal{M}\varphi(|\cdot|^{1/2}) + \mathcal{M}\varphi(|\cdot|^{-1/2}) \;-\; \sum_{|\operatorname{Re}\rho - \frac{1}{2}|<\alpha} N(\rho)\, \mathcal{M}\varphi(\chi|\cdot|^{\rho-1/2}).$$

The left side is Weil's; the right side has only the zeros in the strip. The deduction of the hypothesis, sketched in §1.4, applies the formula to test functions approximating $t \mapsto t^{-z}\mathbf{1}_{t>1}$. The prime-power terms then sum to the logarithmic derivative of the L-function, and the formula shows that $L'/L(s + \frac{1}{2}, \chi)$ is holomorphic on $\operatorname{Re} s > \alpha$ apart from the pole of ζ. Since α is arbitrary, L has no zeros to the right of the critical line, and the functional equation disposes of the left.



The argument and the state of each step in the released draft. Green: written. Amber: written, with open footnotes or sketched proofs. Red: missing, or containing an error that invalidates the step as written.

What Is New

The manuscript is explicit about its novelty, and its claims fall into four groups.

Orbital Schauder bases. The central device is to write each piece of the degenerating kernel in a basis, as one does in linear algebra, and to compute its trace from the diagonal of the resulting matrix. The bases are copies of the exponential system on circles $\mathbb{R}_{>0}/N\mathfrak{p}^{\mathbb{Z}}$, transported to subquotients of the theta space. Masullo calls this "the crux of the matter" [1, §1.4.9] and states that the idea of using bases to reduce the global calculation to a finite one "was never considered in any form nor part, before" [1, §1.5.1].

Lineage. Tate's thesis for the zeta integrals; Weil's explicit formulas; Connes's S-local trace formula (1998–99); Meyer's spectral realization of the explicit formulas (2005) [10, 9, 12, 13]. The step-by-step degeneration is Masullo's alternative to Meyer's infinite Euler-product operators.

Hardy–Sobolev topologies. The weighted spaces $W_I^{s,2}$, with the weight $\max(|g|^{-2\alpha}, |g|^{2\alpha})$, are introduced for this purpose; the manuscript notes that they "were never considered in the literature" [1, §1.1.3]. They play the role of Connes's weighted space L^2_δ, but with a weight chosen so that only zeros in a strip of half-width α are seen. Masullo insists that the topology is chosen entirely by the needs of the geometric side, and that the zeros never enter its choice.

Polar multiplicities and number fields. Theorem 7.1.3.2 shows that Euler factors at places of different residue characteristics share at most one pole, using the $\mathbb{Q}$-linear independence of the logarithms of primes. This controls which coefficient functionals are continuous, and it is where the argument uses something true of number fields and false of function fields. In §14.2 Masullo explains that over a function field the Euler factors share infinitely many poles, all residue fields being powers of one prime, and the method yields only Weil's explicit formula. He offers this as the reason the method proves something new for number fields and nothing new for curves over finite fields.

Deninger's dynamical picture. The supports of the dual basis elements are organized as quotient stacks: periodic orbits $[\mathbb{R}_{>0}/F_v^\times]$ at the primes and fixed points at the archimedean places, as Deninger predicted for a hypothetical dynamical system [11] whose periodic orbits are the primes. Masullo describes this as a confirmation of the correct parts of Deninger's programme, and as logically inessential.

The manuscript also stresses two absences, by analogy with Deligne's proof of the Weil conjectures [14]. Weil's positivity criterion is not used: the test functions vanish on C_F^1, and the only function of the form $\psi * \psi^*$ that does so is zero. And the non-vanishing of $L(s, \chi)$ on the line $\operatorname{Re} s = 1$ is used as an input, much as Deligne first established non-vanishing on the edge of the critical strip.

How It Differs from Alpöge's Proof

The two proofs share a target and almost nothing else. Alpöge's argument bounds a sum over primes from above and lets Landau's theorem turn the bound into a zero-free half-plane. Masullo's establishes an identity and reads the zeros off it. Alpöge's leaves GL_1 at once, for unitary groups, theta lifts and Galois representations; Masullo's never leaves it. Where Alpöge's draws its strength from arithmetic positivity, a nonzero integer having absolute value at least one, Masullo's draws it from the choice of a topology.

	Masullo	**Alpöge**
the core statement	an identity: a trace formula with only strip zeros on the spectral side	an inequality: an upper bound on a smoothed sum over primes
how zeros are excluded	a zero off the line is invisible in the coarse topology	a zero off the line would make the prime sum exceed the bound (Landau)
the group	GL_1 over a number field: abelian, one L-function at a time	unitary groups of higher rank, Shimura varieties, the theta correspondence
the source of leverage	functional analysis: nuclear operators, Schauder bases, the choice of norm	arithmetic positivity: Arakelov slope inequality, forced p-adic divisibility
how primes enter	Weil's local terms, killed one prime at a time by Poisson summation	Chebyshev's divisibility, bounded above by exact periods
families	not used	essential: families of characters, random prime twists, averages
model	Deligne's Weil II, Connes, Deninger	Stepanov's proof for curves over finite fields
scope	all number fields, all Hecke characters	Dirichlet characters over $\mathbb{Q}$
the form of release	an unfinished 940-page manuscript	17.4 million lines of Lean, accepted by its kernel and by independent checkers

The two arguments meet at one point. Each ends by showing that the logarithmic derivative $-L'/L$ continues analytically to a half-plane to the right of the critical line. Alpöge's obtains this from a one-sided bound, through Landau's theorem on Dirichlet series with non-negative coefficients; Masullo's obtains it from an exact formula. The routes into that last step are unrelated. If Alpöge's proof is confirmed, it shows that the Riemann Hypothesis can be reached through arithmetic geometry. It says nothing either way about whether Masullo's route through harmonic analysis on GL_1 can be made to work.

The Errors That Have Been Found

Several specific errors were reported within a day of the release. The first public refutation, and the most widely cited, is Nielsen's note, so it is described first. Its two counterexamples are the first and third errors below. The four errors are in the manuscript as described: the first three were checked against the text for this document, and the fourth is reported in a public log. None of them touches the trace formula itself. Each is either in the deduction of the hypothesis from the trace formula, or in a step whose statement can be corrected without changing the argument.

Nielsen's note

The note [6] is three pages long and concerns the version of the manuscript released on 20 November [6, §1]. Its main claim is that the uniform bound used to apply Montel's theorem in §1.4.3, and again in §1.8.4.3, is false: for the trivial character over $\mathbb{Q}$ the archimedean integrals of the proposed approximants diverge. It concludes that "the application of Montel's theorem ... is invalid" and that "the subsequent deduction of GRH is not established by that argument" [6, Cor. 1]. It adds two points about scope. The objection "applies even if the preceding trace formula is granted," and "a modified test-function family, subtraction, or renormalization would require a new argument ... Such a modification cannot be presumed in assessing the proof as written" [6, §3]. A second proposition shows that the identification of the divisorial strata with a norm preimage is false [6, Prop. 2]. The note ends: "This is a criticism of the proposed proof, not a disproof of GRH" [6, §5].

The errors

1. The uniform bound behind the Montel argument (§1.4.3, p. 29; repeated in §1.8)

To deduce the hypothesis, §1.4.2 approximates $\varphi_z(t) = t^{-z}\mathbf{1}_{t>1}$ by smooth functions $\varphi_n = h_n \cdot (\rho_\varepsilon * \varphi_z)$, where ρ_ε is a mollifier and h_n a bump supported in $(1,\infty)$ increasing to $\mathbf{1}_{(1,\infty)}$. §1.4.3 then asserts that the archimedean terms

$$A_n(z) = \frac{1}{2}\int_{\mathbb{R}_{>0}} \sum_{u=\pm 1} \frac{\varphi_n(tu)\, t^{1/2}}{|1-tu|}\,\frac{dt}{t}$$

are bounded by a constant independent of z and of n, so that Montel's theorem extracts a holomorphic limit. They are not bounded. Take $F = \mathbb{Q}$, the trivial character and $z = 2$. The approximants are non-negative and converge to t^{-2} on $t > 1$, so Fatou's lemma gives

$$\liminf_{n\to\infty} A_n(2) \;\geq\; \frac{1}{2}\int_1^2 \frac{t^{-5/2}}{t-1}\,dt \;=\; +\infty.$$

The limit function jumps at $t = 1$, where the archimedean kernel $1/|1-t|$ is not integrable. Weil's principal value cancels this singularity only for test functions that are continuous across $t = 1$, and these one-sided approximants are not. This is Proposition 1 of Nielsen's note [6, Prop. 1], also reported in the public formalization repository [7], and it is confirmed here.

Severity. Invalidates §1.4.3, and the deduction of the hypothesis as written. Nielsen is right that a repair is a new argument and cannot be presumed: a renormalized family of test functions, for instance, would have to be justified term by term in the trace formula. The repair is nevertheless short and standard, and it does

not need Montel's theorem at all. Once the trace formula holds for every smooth compactly supported test function vanishing on C_F^1, the hypothesis follows from it by the argument in the box below, which is the content of Connes's observation [12] that such a trace formula is equivalent to the hypothesis. Nielsen's note shows that the manuscript does not supply this deduction. It does not show that the deduction is unavailable, and the note does not claim that it is.

Deducing the hypothesis without Montel's theorem

Suppose the trace formula of Theorem 12.5.1.1 holds for every smooth compactly supported φ on C_F that vanishes on C_F^1. Choose $\alpha > 0$ different from every $|\operatorname{Re}\rho - \frac{1}{2}|$, so that no zero lies on the edge of the strip. Weil's explicit formula holds for the same test functions with the same left side, and subtracting the two formulas gives

$$\sum_{|\operatorname{Re}\rho-\frac{1}{2}|>\alpha} N(\rho)\,\mathcal{M}\varphi(\chi|\cdot|^{\rho-1/2}) = 0.$$

Take $\varphi(g) = \overline{\chi}(g)\,u(\log|g|)$ with u smooth and supported in $(0,\infty)$. Up to a constant, $\mathcal{M}\varphi(\chi|\cdot|^{\rho-1/2}) = \hat{u}(\rho-\frac{1}{2})$ with $\hat{u}(w) = \int u(x)e^{wx}\,dx$, and the sum converges absolutely because $\hat{u}$ decays rapidly on vertical strips. So the exponential sum $S(x) = \sum N(\rho)e^{(\rho-1/2)x}$ over the zeros outside the strip vanishes, as a distribution, on $x > 0$.
Now take $u \neq 0$ supported in $(0,\varepsilon)$. The convolution $(S * u)(x) = \sum N(\rho)\,\hat{u}(\frac{1}{2}-\rho)\,e^{(\rho-1/2)x}$ vanishes for $x > \varepsilon$, so its Laplace transform $\int_0^\infty (S*u)(x)e^{-wx}\,dx$ is entire. For $\operatorname{Re} w > \frac{1}{2}$ that transform equals

$$\sum N(\rho)\,\frac{\hat{u}(\frac{1}{2}-\rho)}{w-(\rho-\frac{1}{2})},$$

which converges absolutely away from its poles and is meromorphic, with a simple pole at each $\rho-\frac{1}{2}$ of residue $N(\rho)\,\hat{u}(\frac{1}{2}-\rho)$. Since $\hat{u}$ is entire and not identically zero, u can be chosen so that this residue is nonzero at any given zero. Hence there are no zeros outside the strip, and since α is arbitrary, none off the critical line.

2. A sign in the logarithmic derivative (§1.4.4, p. 29)

With $\varphi(t) = t^{-z}$ for $t > 1$, the prime-power terms are

$$\sum_v \log q_v \sum_{m\geq 1} \chi(\pi_v^m)\,q_v^{-m/2}\,q_v^{-mz} \;=\; -\frac{L_f'}{L_f}(z+\tfrac{1}{2},\chi),$$

with a minus sign. §1.4.4 prints $+L_f'/L_f$. The error was reported in the public formalization repository [7] and is confirmed here.

Severity. A misprint. It does not affect the conclusion, which concerns where L'/L is holomorphic, not its sign.

3. The strata as a norm preimage (§1.3.2 and §11.3.1.1, p. 493)

The manuscript defines the stratum $\mathrm{GL}_1(\mathbb{A}_F)_{P_k}$ as the set of ideles whose components at every place above the first k rational primes are units. It then asserts that this stratum is the preimage, under the norm map to $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$, of the corresponding stratum over $\mathbb{Q}$. That is false whenever a prime splits. In $F = \mathbb{Q}(i)$ the prime 5 splits into two places v_1, v_2 of degree

one. An idele with component π_{v_1} at v_1, $\pi_{v_2}^{-1}$ at v_2 and units elsewhere has norm a unit at 5, so it lies in the preimage, but its components at v_1 and v_2 are not units, so it is not in the stratum. The norm preimage only sees $\sum_{v|p} f_v \operatorname{ord}_v(g_v) = 0$. This is Proposition 2 of Nielsen's note [6, Prop. 2], which uses the same idele, and it is confirmed here.

Severity. Harmless. The manuscript gives the correct explicit description immediately afterwards ("Explicitly, we have ..."), and the argument uses that description, not the norm preimage.

4. Reported: stabilizers and a local term at the prime 2 (Prop. 8.2.0.4, p. 386; §11.3.2)

Two further errors are reported but were not checked here. The first is in Prop. 8.2.0.4, which asserts that a transporter between points of the stack-theoretic orbits is unique modulo the local unit group; a counterexample is reported in the public formalization repository [7], with the suggested repair of restricting to norm-one ideles. The second is in an independent research log [8]: there, a lift in the orbital trace computation is said to give 0 at the prime 2 where the local term should be $\log 2$.

Severity. Both concern normalizations inside steps whose overall shape is unaffected. The first matters to the stack-theoretic picture, which the manuscript itself describes as logically inessential. The second, if confirmed, would need a corrected normalization of the first shell of the filtration, which the same log suggests can be done.

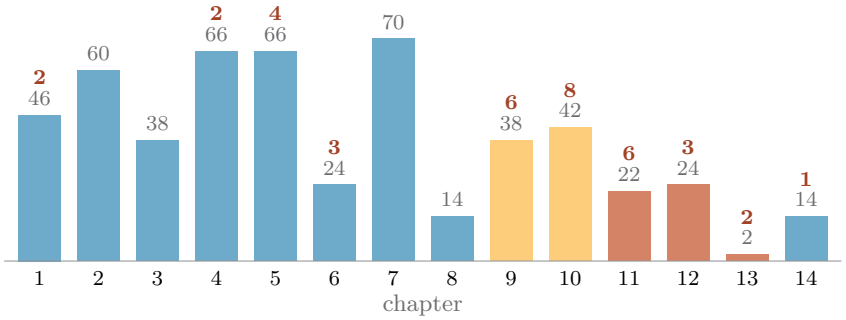
Nielsen's note, assessed

The note is correct, and it is careful. Both counterexamples hold, and its conclusions are stated at exactly the strength its counterexamples support: the Montel argument is invalid as written, the deduction of the hypothesis is not established by it, and the note is "a criticism of the proposed proof, not a disproof of GRH" [6, §5]. Its insistence that the objection holds even if the trace formula is granted, and that a repair cannot be presumed, is the right standard for a proof as written. The manuscript as released does not prove the Generalized Riemann Hypothesis, even granting everything before §1.4.3.

Its limits are those of its scope. The note addresses the deduction of the hypothesis from the trace formula, which is the direction that can be repaired, as the box above shows. It does not address the trace formula itself, which is equivalent to the hypothesis and whose proof runs through the unwritten §12.3. It therefore shows that the manuscript, as released, contains an invalid step in its final argument. It does not show that the strategy fails, and it does not claim to. Readers who took the note as a refutation of Masullo's approach read more into it than it says.

What Is Structurally Missing

The more serious issue is not any error but what the draft does not yet contain. The manuscript has 536 pages of main text and 396 of appendices. Its unfinished parts are concentrated exactly where the argument has to come together: the last three chapters before the endgame, which are also the shortest.



grey numbers: pages
red numbers: footnotes such as "Return here," "Include section" or "Correct statement"

Pages per chapter of the main text, with the number of the author's own unfinished-work footnotes above each bar. Chapters 9 to 13 contain twenty-five of the forty such footnotes. Chapter 13, where the zeros are finally located, is two pages long.

Location	State in the released draft
§12.3, global to S-local deconstruction	A heading and the footnote "Include section." This is the step that assembles the orbital pieces and the S-local remnant into the global trace.
Chapter 13, zeros of Hecke L-functions	Two pages. §13.1 and §13.2 are headings only. The trace formula is restated with the footnote "Correct statement"; the divisor theorem reads "Insert statement," and refers to "Theorem ??."
Theorem 11.1.2.1, the shell structure	Proved for the unweighted Sobolev space, on the half-line $\lvert g\rvert \geq 1/\Lambda$. Part (2) contains two "Return here" footnotes in place of steps.
Theorem 11.3.2.1, orbital traces	A proof of about one page, which appeals to "the same method used in the estimates from Theorem 10.3.2.1."
Theorem 12.4.1.1, S-local trace	Written; a footnote records that the cutoff functions of Definition 9.3.3.1 must be extended before the argument applies as stated.
Introduction	§§1.2.4–1.2.6 and §§1.6.1–1.6.2 are headings only; §1.3.3 and §1.3.4 end in "Finish!"

Missing exposition is expected in a draft released early, and much of this is the "filling in some writing" the acknowledgements describe. Two of the gaps are different in kind. Chapter 13 is the deduction of the zeros from the trace formula. As the previous section explains, that deduction can be made by other means. §12.3 is the proof of the trace formula. Nothing else in the manuscript can stand in for it.

Where the Real Difficulty Lies

The comparison that must fail if the hypothesis fails

The cleanest way to see what §12.3 must do is to run the same argument in two topologies. In Meyer's fine topology, the theta space, its filtration by primes, the orbital pieces and the S-local remnant all make sense. The same calculation of the geometric side goes through, and its spectral side contains every zero. In Masullo's coarse topology the spectral side contains only the zeros in the strip. The local calculations that produce the geometric side, the orbital traces and Connes's S-local trace, are calculations with the convolution kernel $\varphi(xy^{-1})$, and that kernel does not depend on the topology.

So if there were a zero off the line, the two computations would have to differ somewhere, and the only place they can differ is in how the pieces are assembled: in §12.3, and in the limits feeding into it. Any complete proof must show that, in the coarse topology, the assembly produces the same total as in the fine one. Showing that is equivalent to the hypothesis. This does not make the strategy circular, but it locates the entire difficulty in the section that is not yet written.

Three limits and the spectral kernel

On a reading of the draft, three points in the assembly need an argument that the text does not yet provide.

- **The order of the limits.** The geometric side involves three limits: the filtration index $k \to \infty$, the truncation $\Lambda \to \infty$, and the sum over primes. The shell-structure convergence of Theorem 11.1.2.1 is proved with the truncation in place, on the half-line $|g| \geq 1/\Lambda$, where removing Euler factors one at a time is controlled by dominated convergence. The coarse topology is designed so that infinite Euler products do not converge in it. That is Mertens' obstruction, which Masullo himself names, and an interchange of these limits is where it would reappear.
- **The space in which the convergence is proved.** Theorem 11.1.2.1 is stated for the unweighted space $W^{s,2}(C_F)$, the "0-line" of its title. Theorem 11.3.2.1 invokes it for the weighted space $W^s_I(C_F)$ with $I = [-\alpha, \alpha]$. The weight $\max(|g|^{-2\alpha}, |g|^{2\alpha})$ is unbounded at both ends, and convergence in the unweighted norm on a half-line does not imply convergence in the weighted norm on the whole line.
- **The spectral kernel's share.** Each orbital piece is $(\alpha_{k+1} - \alpha_k)k_{\text{geom}}$, and k_{geom} is the convolution kernel *minus* the spectral kernel. The proof of Theorem 11.3.2.1 evaluates each piece through the action of convolution on the orbital basis. It does not show what becomes of the corresponding pieces of the spectral kernel. Their diagonal integrals add up to the whole spectral side, which is not zero, so their distribution among the pieces is precisely the information the assembly must control.

None of these is shown here to be false. Each is a place where a revision has to supply an argument, and the comparison with the fine topology says that at least one of them carries the full weight of the Riemann Hypothesis.

A claim stronger than the hypothesis

The manuscript contains a second claim, and its status is inconsistent across the draft. In the letter to Sarnak [1, §§1.7.2–1.7.3], Masullo explains that the extreme case $\alpha = 0$ of the method "does not lead to a Trace Formula anymore." The reason he gives is that algebraic Hecke characters attached to CM abelian varieties can have central zeros of multiplicity greater than $[F : \mathbb{Q}]$, which the formula would forbid. The multiplicity bound that the case would give is therefore presented as an *expectation* and "reserved for sequel work."

Chapter 13, however, states as a theorem a trace formula on the critical line itself, with each zero counted with multiplicity $\min\{N(\rho), [F : \mathbb{Q}]\}$, for test functions satisfying $[F : \mathbb{Q}]$ vanishing conditions. Chapter 14 refers to "the Multiplicity Bound obtained in Theorem 11.2.2.1." Compared with Weil's formula, this version would say that, away from the points excluded by the vanishing conditions, every zero has multiplicity at most $[F : \mathbb{Q}]$. For $F = \mathbb{Q}$ that is the simplicity of the zeros of every Dirichlet L-function, a famous open problem believed to be deeper than the Riemann Hypothesis itself. An independent research log [8] reports that a Hecke L-function with a zero of order at least three contradicts the bound as stated. Whether the example survives the vanishing conditions depends on their exact form, and that is the statement the author has marked "Correct statement."

The Generalized Riemann Hypothesis does not depend on this claim; the trace formula in a strip of positive width is enough. But a method whose stated output exceeds what is believed provable, and in one formulation contradicts known examples, calls for particular care with the identity it rests on.

No test case

A new method is usually checked first on a case where the answer is known. For this method there is none. Over function fields, where the Riemann Hypothesis is Weil's theorem, Masullo shows in §14.2 that the method cannot be carried out: it collapses to Weil's explicit formula, which carries no information about the zeros. Over number fields, the answer is the open problem itself. This is not an objection to the argument, and Masullo gives a coherent reason for the difference: the archimedean places, and the independence of the logarithms of distinct primes. It does mean that the method's central mechanism has never been tested against a known result, and that its first verification will have to be a complete proof.

Assessment

How critical are the known errors?

Not very. The Montel error is real: the uniform bound in §1.4.3 fails, by a one-line Fatou argument. But it sits in the deduction of the hypothesis from the trace formula, and that deduction is the easy direction. The sign error and the norm-preimage error are slips, and the remaining reported errors concern normalizations in steps whose shape is sound. A revision can fix all of them without changing the strategy. Treating the Montel failure as the refutation of the manuscript mistakes a broken argument for a step with a sound one in its place.

Was Masullo on the right track?

The programme is serious and the central idea is genuinely new. Connes's spectral interpretation is one of the most studied approaches to the Riemann Hypothesis. Meyer showed that its trace calculation can be carried out rigorously, at the cost of a topology that sees every zero. Masullo's proposal, to carry out the degeneration from global to local one prime at a time, with exact bases on each step, in a topology coarse enough to hide the off-line zeros, is a precise and original answer to the obstruction that stopped earlier work. The supporting machinery of the first ten chapters is careful, and Conrad's months of reading lend it weight.

What has not been shown is that the idea works. The step at which it has to overcome the divergence of Euler products in the critical strip is the assembly of §12.3. That section is empty, and the convergence results that feed it are proved in a weaker setting than the one in which they are used. Every previous attempt along these lines has foundered at exactly this point, because a trace formula of this shape is equivalent to the hypothesis. So the question that matters is not whether the known errors are fatal. It is whether the coarse-topology assembly can be proved at all, and the released draft does not yet contain the material to decide it.

Is it the same method as Alpöge's?

No. The two proofs share the conclusion and the last analytic step, the continuation of L'/L to a half-plane, and nothing upstream of it. Alpöge's proof is a one-sided inequality built from arithmetic positivity on higher-rank groups; Masullo's is an exact identity built from functional analysis on GL_1. A confirmation of one would not transfer to the other. If both survived, it would mean two genuinely different explanations of why the zeros lie on the line: one in the language of Chebyshev and Arakelov, the other in the language of Weil and Connes.

References

[1] A. M. Masullo, *The Music of the Primes*, manuscript, 940 pp., released 20 November 2026, https://math216.wordpress.com/wp-content/uploads/2026/11/the-music-of-the-primes.pdf.

[2] R. Vakil, "Hurried Announcement: Alessandro Masullo GRH," *Math 216: Foundations of Algebraic Geometry* (blog), 20 November 2026, with an update the same day, https://math216.wordpress.com/2026/11/20/hurried-announcement-alessandro-masullo-grh/.

[3] L. Alpöge, *Remark*, note accompanying the Lean source of the formalization, 20 November 2026, https://alpo.ge/remark.pdf.

[4] Anthropic, announcement of a Claude-assisted bound on the proportion of zeros of the Riemann zeta function on the critical line, 10 August 2026, https://www.anthropic.com/news/riemann-zeta.

[5] T. Fernholz, "AI claimed a solution to the 167-year-old Riemann hypothesis — and a human did, too," *TechCrunch*, 20 November 2026, https://techcrunch.com/2026/11/20/ai-claimed-a-solution-to-the-167-year-old-riemann-hypothesis-and-a-human-did-too/.

[6] J. L. Nielsen, "A Failure of Uniform Boundedness in Masullo's Proposed Proof of the Generalized Riemann Hypothesis," preprint, November 2026, 3 pp., https://philarchive.org/rec/NIEIOM.

[7] "Scout report: existing formalizations across The Music of the Primes," issue 3, mccorvie/riemann-masullo, GitHub, 20 November 2026, https://github.com/mccorvie/riemann-masullo/issues/3.

[8] *The Music of the Primes: Revise*, independent research log on the manuscript, GitHub, November 2026, https://github.com/Titulus233/The-Music-of-the-Primes-Revise.

[9] A. Weil, "Sur les 'formules explicites' de la théorie des nombres premiers," *Comm. Sém. Math. Univ. Lund*, tome supplémentaire (1952), 252–265.

[10] J. T. Tate, "Fourier analysis in number fields and Hecke's zeta-functions," in J. W. S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Academic Press, 1967, 305–347.

[11] C. Deninger, "Lefschetz trace formulas and explicit formulas in analytic number theory," *J. reine angew. Math.* 441 (1993), 1–15.

[12] A. Connes, "Trace formula in noncommutative geometry and the zeros of the Riemann zeta function," *Selecta Math. (N.S.)* 5 (1999), 29–106.

[13] R. Meyer, "On a representation of the idele class group related to primes and zeros of L-functions," *Duke Math. J.* 127 (2005), 519–595.

[14] P. Deligne, "La conjecture de Weil. II," *Publ. Math. IHÉS* 52 (1980), 137–252.